Basic Calculations Of The Wave Structure Of Matter Theory: Particle & Anti-Particle Waves

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An outline of this material is provided in Milo Wolff's book Exploring The Physics Of The Unknown Universe, 2nd edition, 1994, pp. 238-245. Any errors in these calculations are my own.

Consider the linear, homogeneous wave equation:

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0 \tag{1}$$

In this equation ψ is the quantum wave's complex amplitude.

In spherical polar coordinates, the Laplacian can be written as

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2} \tag{2}$$

If we **assume** that the wave function ψ is spherically symmetric then it doesn't depend on θ or φ and the wave equation reduces to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \psi \right) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0 \tag{3}$$

$$\frac{1}{r^2} \left(2r \frac{\partial}{\partial r} \psi + r^2 \frac{\partial^2}{\partial r^2} \psi \right) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0 \tag{4}$$

$$\left(\frac{\partial^2}{\partial r^2} \psi + \frac{2}{r} \frac{\partial}{\partial r} \psi \right) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi \tag{5}$$

where $\psi = \psi(r, t)$.

Using the method of separation of variables, we write the wave function as

$$\psi(r, t) = A(r) B(t) \tag{6}$$

We then get

$$B(t)\left(\frac{\partial^2}{\partial r^2}A(r)\frac{2}{r}\frac{\partial}{\partial r}A(r)+\frac{2}{r}\frac{\partial}{\partial r}A(r)\right)=A(r)\frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}B(t) \tag{7}$$

$$\frac{1}{A(r)}\left(\frac{\partial^2}{\partial r^2}A(r)+\frac{2}{r}\frac{\partial}{\partial r}A(r)\right)=\frac{1}{\mathrm{c}^2}\frac{1}{B(t)}\frac{\partial^2}{\partial t^2}B(t) \tag{8}$$

The left hand side depends only on the spatial variable (r) while the right hand side depends only on the time variable (t). Because each side of the equation is independent of the other, both sides must be equal to the same constant. The choice we use here $(-k^2)$ allows for oscillatory solutions.

The spatial dependent equation (left hand side of equation (8)) becomes

$$\frac{1}{A(r)}\left(\frac{\partial^2}{\partial r^2}A(r)+\frac{2}{r}\frac{\partial}{\partial r}A(r)\right)=-k^2 \tag{9}$$

$$\frac{\partial^2}{\partial r^2}A(r)+\frac{2}{r}\frac{\partial}{\partial r}A(r)=-k^2\,A(r) \tag{10}$$

$$r\frac{\partial^2}{\partial r^2}A(r)+2\frac{\partial}{\partial r}A(r)=-k^2\,r\,A(r) \tag{11}$$

We now do a change of variables by defining the function

$$u(r)=r\,A(r) \tag{12}$$

Then

$$\frac{\partial}{\partial r}u(r)=A(r)+r\frac{\partial}{\partial r}A(r) \tag{13}$$

and

$$\frac{\partial^2}{\partial^2 r}u(r)=\frac{\partial}{\partial r}A(r)+\frac{\partial}{\partial r}A(r)+r\frac{\partial^2}{\partial r^2}A(r) \tag{14}$$

$$\frac{\partial^2}{\partial^2 r}u(r)=2\frac{\partial}{\partial r}A(r)+r\frac{\partial^2}{\partial r^2}A(r) \tag{15}$$

This (15) is the same as the left hand side of equation (11). So we now have

$$\frac{\partial^2}{\partial r^2} u(r) = -k^2 \, u(r) \tag{16}$$

This has the solutions

$$u(r) = g_1 \, e^{ikr} + g_2 \, e^{-ikr} \tag{17}$$

where g_1 and g_2 are arbitrary constants.

We therefore have (using (12))

$$A(r) = g_1 \, \frac{e^{ikr}}{r} + g_2 \, \frac{e^{-ikr}}{r} \tag{18}$$

The time dependent equation is (right hand side of equation (8))

$$\frac{1}{\mathrm{c}^2} \frac{1}{B(t)} \frac{\partial^2}{\partial t^2} B(t) = -k^2 \tag{19}$$

$$\frac{\partial^2}{\partial t^2} B(t) = -k^2 \, \mathrm{c}^2 \, B(t) \tag{20}$$

This equation has the solutions

$$B(t) = g_3 \, e^{ikct} + g_4 \, e^{-ikct} \tag{21}$$

Then, assembling the spatial and time equations back together gives

$$\psi(r,t) = \left(g_1 \, \frac{e^{ikr}}{r} + g_2 \, \frac{e^{-ikr}}{r} \right) \left(g_3 \, e^{ikct} + g_4 \, e^{-ikct} \right) \tag{22}$$

$$\psi(r,t) = g_1 \, g_3 \, \frac{e^{ikr} e^{ikct}}{r} + g_1 \, g_4 \, \frac{e^{ikr} e^{-ikct}}{r} + g_2 g_3 \, \frac{e^{-ikr} e^{ikct}}{r} + g_2 g_4 \, \frac{e^{-ikr} e^{-ikct}}{r} \tag{23}$$

Each of these terms is a solution to the original wave equation (1).

$$\psi_1(r,t) = g_1\ g_3 \frac{e^{ikr}e^{ikct}}{r} = A_1 \frac{e^{(ik_1ct+ik_1r)}}{r} \tag{24}$$

$$\psi_2(r,t) = g_1\ g_4 \frac{e^{ikr}e^{-ikct}}{r} = A_2 \frac{e^{(-ik_2ct+ik_2r)}}{r} \tag{25}$$

$$\psi_3(r,t) = g_2 g_3 \frac{e^{-ikr}e^{ikct}}{r} = A_3 \frac{e^{(ik_3ct-ik_3r)}}{r} \tag{26}$$

$$\psi_4(r,t) = g_2 g_4 \frac{e^{-ikr}e^{-ikct}}{r} = A_4 \frac{e^{(-ik_4ct-ik_4r)}}{r} \tag{27}$$

We interpret k as being the inverse of the wavelength $k = \frac{2\pi}{\lambda}$. This gives $kc = \frac{2\pi c}{\lambda} = \omega$, the angular frequency of the wave.

In the WSM theory, a particle, e.g. the electron, positron, proton, antiproton, is composed of an inward traveling wave and an outward traveling wave to form a standing wave pattern. We can combine the above wave functions as follows.

$$\psi_P(r,t) = \psi_{1-IN} - \psi_{3-OUT} = A_1 \frac{e^{(ik_1ct+ik_1r)}}{r} - A_3 \frac{e^{(ik_3ct-ik_3r)}}{r} \tag{28}$$

$$\psi_A(r,t) = -\psi_{2-IN} + \psi_{4-OUT} = -A_2 \frac{e^{(-ik_2ct+ik_2r)}}{r} + A_4 \frac{e^{(-ik_4ct-ik_4r)}}{r} \tag{29}$$

where "*P*" and "*A*" indicate particle and anti-particle and the subscripts "*IN*" and "*OUT*" indicate inward and outward traveling waves.

For the case that there is **no force** on the particles and as seen from their **rest frame** of reference, we set $A_1 = A_3$, $k_1 = k_3$ and $A_2 = A_4$, $k_2 = k_4$ to get

$$\psi_P(r,t) = A_P \frac{e^{(ikct+ikr)} - e^{(ikct-ikr)}}{r} \tag{30}$$

$$\psi_P(r,t) = A_P \frac{\left(e^{ikr} - e^{-ikr}\right)e^{ikct}}{r} \tag{31}$$

$$\psi_A(r,t) = -A_A \frac{e^{(-ikct+ikr)} - e^{(-ikct-ikr)}}{r} \tag{32}$$

$$\psi_A(r,t) = -A_A \frac{\left(e^{ikr} - e^{-ikr}\right)e^{-ikct}}{r} \tag{33}$$

Now using the equation

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{34}$$

we have

$$\psi_P(r,t) = A_P \frac{\sin(kr)\, ie^{ikct}}{r} \tag{35}$$

$$\psi_A(r,t) = A_A \frac{-\sin(kr)\, ie^{-ikct}}{r} \tag{36}$$

(We have absorbed the factor of 2 into the *A* constants.)

Note that as $r \to 0$ the factor $\dfrac{\sin(kr)}{r}$ and hence the wave amplitude remains finite. This can be seen by using the expansion

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} \tag{37}$$

Then

$$\lim_{r\to 0}\frac{\sin(kr)}{r}=\lim_{r\to 0}\ \frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\frac{\left(kr\right)^{(2n+1)}}{(2n+1)!} \tag{38}$$

$$=\lim_{r\to 0}\left(1-\frac{k^3r^2}{3!}+\frac{k^5r^4}{5!}-\cdots\right)=1 \tag{39}$$

This ensures that all waves have a finite amplitude at the "particle's" center.

For the electron, with $k=\frac{2\pi}{\lambda}$, $kc=\frac{2\pi c}{\lambda}=\omega$

$$\psi_e\left(r,t\right)=\ A\frac{\sin\left(\frac{2\pi r}{\lambda}\right)ie^{i\omega t}}{r} \tag{40}$$

The wavelength λ is to be interpreted as the Compton wavelength of the particle.

(From: http://en.wikipedia.org/wiki/Compton_wavelength):

The Compton wavelength is given by

$$\lambda=\frac{h}{mc} \tag{41}$$

where *h* is the Planck constant, *m* is the particle's rest mass and *c* is the speed of light. For the electron, the Compton wavelength has been deduced to be

$$\lambda_e\simeq 2.4263102175\times 10^{-12}\ \text{meters} \tag{42}$$

We can also rewrite the equations as

$$\psi_A\left(r,t\right)=\left[A_P\frac{\sin(kr)}{r}\right]ie^{i\omega t}=\left[A_P\frac{\sin(kr)}{r}\right]\left(i\cos(\omega t)-\sin(\omega t)\right) \tag{43}$$

$$\psi_P\left(r,t\right)=\left[A_A\frac{-\sin(kr)}{r}\right]ie^{-i\omega t}=\left[A_A\frac{-\sin(kr)}{r}\right]\left(i\cos(\omega t)+\sin(\omega t)\right) \tag{44}$$

If we keep only the real part of the wave functions then we have

$$\psi_{P-REAL}(r,t) = A_P \frac{\sin(kr)}{r} \sin(\omega t) \tag{45}$$

$$\psi_{A-REAL}(r,t) = A_A \frac{-\sin(kr)}{r} \sin(\omega t) \tag{46}$$

(A function of the form $f(x) = \frac{\sin(x)}{x}$ is called an unnormalized sinc function.)

In this case (real part of the complex particle wave function only) we note that the particle wave cancels the antiparticle wave when they have the same maximum amplitude and are at the same position.

$$\psi_{P-REAL}(r,t) + \psi_{A-REAL}(r,t) = 0 \tag{47}$$

Thus the particle and anti-particle annihilate each other. This does not happen if the complex form of the waves (31) and (33) are used.

A standing wave can be decomposed into two traveling waves in opposite directions.

$$\sin(kr)\sin(\omega t) = \frac{\cos(kr-\omega t)}{2} - \frac{\cos(kr+\omega t)}{2} \tag{48}$$

So

$$\psi_P(r,t) = A_P \frac{\sin(kr)}{r} \sin(\omega t) = \frac{A_P}{2r}\left(\cos(kr-\omega t) - \cos(kr+\omega t)\right) \tag{49}$$

$$\psi_A(r,t) = A_A \frac{-\sin(kr)}{r} \sin(\omega t) = \frac{A_A}{2r}\left(\cos(kr+\omega t) - \cos(kr-\omega t)\right) \tag{50}$$

Note that if we replace $t \to -t$ (and assuming equal amplitudes) then the particle wave function becomes the anti-particle wave function and visa-versa. This does not happen with the complex wave functions.

If we replace $x \to -x$, $y \to -y$, and $z \to -z$ (a parity transformation) on the "particle" wave function, it remains the "particle" wave function. Similarly for the "anti-particle" wave function. That is, they are functions of $r = \sqrt{x^2 + y^2 + z^2}$ which does not change sign under a parity transformation.

We need to be careful when selecting which wave function to use in a particular situation. The equations (31) and (33) and subsequent equations are only valid for the case of no interactions (no force acting on the particles).

The WSM theory also uses a Universal Time to achieve a Universal Synchronization so that all like-charged particles have the same phase φ_+ at the particles' center, ***<u>everywhere and at the same time</u>***, while all the opposite charged particles have a phase $\varphi_- = \varphi_+ + 180^\circ$, ***<u>everywhere and at the same time</u>***.

