re 6-16. Gauss's law applies to a ing charge: the surface integral of aluated over the fixed sphere gives , as if the charge were stationary. excess E near $\theta = \pi/2$ is just pensated by the weak field near 0 and $\theta = \pi$. See Figures 6-9 and .

by the surface S, and using the form of Gauss's law:

(6-117)

the surface S.
law by integrating the flux of E encloses the charge Q at a given ne 1. The charge passes through 6.
osing t equal to zero. Then

$r^2 \sin\theta \, d\theta$ … $- \beta^2 \sin^2\theta)^{3/2}$, (6-118)

$\sin\theta \, d\theta$ … $^2\theta + (1/\gamma^2)]^{3/2}$, (6-119)

dx … $+ (1/\gamma^2)]^{3/2}$, (6-120)

(6-121)

E of point charges moving at

, when we checked our transforma- ate capacitor, we used Gauss's law E_1 to E_2. We were therefore assum- ally well to stationary and mobile

6.9 THE DIVERGENCE OF $\boldsymbol{B}$

The divergence of $\boldsymbol{B}$ is quite easy to calculate from Eq. 6-34:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{\mu_0\gamma Q\mathcal{V}}{4\pi}\left\{\frac{\partial}{\partial y}\left(\frac{-z}{[\gamma^2(x-\mathcal{V}t)^2+y^2+z^2]^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{y}{[\gamma^2(x-\mathcal{V}t)^2+y^2+z^2]^{3/2}}\right)\right\} \tag{6-122}$$

$$= \frac{\mu_0\gamma Q\mathcal{V}}{4\pi}\left\{\frac{3yz}{[\quad]^{5/2}} - \frac{3yz}{[\quad]^{5/2}}\right\} = 0, \tag{6-123}$$

where we have set

$$[\quad] \equiv [\gamma^2(x-\mathcal{V}t)^2+y^2+z^2]. \tag{6-124}$$

This is the second of Maxwell's equations:

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0} \tag{6-125}$$

or, using the divergence theorem,

$$\boxed{\int_S \boldsymbol{B}\cdot\boldsymbol{da} = 0.} \tag{6-126}$$

Intuitively, it is quite obvious that the divergence of $\boldsymbol{B}$ should be zero: since the lines of $\boldsymbol{B}$ are circles centered on the path of the moving charge, which is the x-axis in Figures 6-9 and 6-10, the total flux of $\boldsymbol{B}$ flowing out of any imaginary volume must be zero. This is a general result: the divergence of $\boldsymbol{B}$ is always zero.

6.10 THE CURL OF $\boldsymbol{E}$

There is a third Maxwell equation, which is stated as follows:

$$\boxed{\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t},} \tag{6-127}$$

or, using Stokes's theorem,

$$\boxed{\oint_C \boldsymbol{E}\cdot\boldsymbol{dl} = -\frac{\partial}{\partial t}\int_S \boldsymbol{B}\cdot\boldsymbol{da} = -\frac{\partial\Phi}{\partial t},} \tag{6-128}$$

From: Lorrain & Corson, Electromagnetic Fields And Waves, 2nd ed., 1970, W. H. Freeman and Company

where S is any surface bounded by the curve C, and where Φ is the magnetic flux linking the curve C.

Let us verify Eq. 6-127 by substituting the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ from Eqs. 6-33 and 6-34:

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}, \tag{6-129}$$

$$(\nabla \times \boldsymbol{E})_x = \frac{\gamma Q}{4\pi\epsilon_0} \left\{ \frac{\partial}{\partial y} \frac{z}{[\quad]^{3/2}} - \frac{\partial}{\partial z} \frac{y}{[\quad]^{3/2}} \right\} = 0, \tag{6-130}$$

$$(\nabla \times \boldsymbol{E})_y = \frac{\gamma Q}{4\pi\epsilon_0} \left\{ \frac{\partial}{\partial z} \frac{x - \mathcal{V}t}{[\quad]^{3/2}} - \frac{\partial}{\partial x} \frac{z}{[\quad]^{3/2}} \right\}, \tag{6-131}$$

$$= \frac{\gamma Q}{4\pi\epsilon_0} \left\{ -\frac{3(x - \mathcal{V}t)z}{[\quad]^{5/2}} + \frac{3\gamma^2(x - \mathcal{V}t)z}{[\quad]^{5/2}} \right\}, \tag{6-132}$$

$$= \frac{3\gamma Q}{4\pi\epsilon_0} \left\{ \frac{(x - \mathcal{V}t)}{[\quad]^{5/2}} (\gamma^2 - 1)z \right\}, \tag{6-133}$$

$$(\nabla \times \boldsymbol{E})_z = \frac{\gamma Q}{4\pi\epsilon_0} \left\{ \frac{\partial}{\partial x} \frac{y}{[\quad]^{3/2}} - \frac{\partial}{\partial y} \frac{x - \mathcal{V}t}{[\quad]^{3/2}} \right\}, \tag{6-134}$$

$$= \frac{\gamma Q}{4\pi\epsilon_0} \left\{ -\frac{3\gamma^2(x - \mathcal{V}t)y}{[\quad]^{5/2}} + \frac{3(x - \mathcal{V}t)y}{[\quad]^{5/2}} \right\}, \tag{6-135}$$

$$= \frac{3\gamma Q}{4\pi\epsilon_0} \left\{ \frac{(x - \mathcal{V}t)}{[\quad]^{5/2}} (1 - \gamma^2)y \right\}, \tag{6-136}$$

while

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\mu_0 \gamma Q \mathcal{V}}{4\pi} \frac{3\gamma^2 \mathcal{V}(x - \mathcal{V}t)}{[\quad]^{5/2}} (-z\boldsymbol{j} + y\boldsymbol{k}). \tag{6-137}$$

We can see that Eq. 6-127 is satisfied since

$$1 - \gamma^2 = -\beta^2\gamma^2 = -(\mathcal{V}/c)^2\gamma^2. \tag{6-138}$$

This is again a general result: the curl of $\boldsymbol{E}$ is always equal to minus the time derivative of $\boldsymbol{B}$. You should be able to show that this is in agreement with the expression for $\boldsymbol{E}$ in terms of V and $\boldsymbol{A}$, which we found in Section 6.5.

6.11 THE CURL OF $\boldsymbol{B}$

The fourth and last of Maxwell's equations is the following:

C, and where Φ is the magnetic

g the values of E and B from

(6-129)

$$\frac{\partial}{\partial z}\frac{y}{[\quad]^{3/2}}\bigg\} = 0, \qquad (6\text{-}130)$$

$$\frac{\partial}{\partial x}\frac{z}{[\quad]^{3/2}}\bigg\}, \qquad (6\text{-}131)$$

$$+\frac{3\gamma^2(x - \mathcal{V}t)z}{[\quad]^{5/2}}\bigg\}, \qquad (6\text{-}132)$$

$$- 1)z\bigg\}, \qquad (6\text{-}133)$$

$$\frac{\partial}{\partial y}\frac{x - \mathcal{V}t}{[\quad]^{3/2}}\bigg\}, \qquad (6\text{-}134)$$

$$t)y + \frac{3(x - \mathcal{V}t)y}{[\quad]^{5/2}}\bigg\}, \qquad (6\text{-}135)$$

$$- \gamma^2)y\bigg\}, \qquad (6\text{-}136)$$

$$(-z\boldsymbol{j} + y\boldsymbol{k}). \qquad (6\text{-}137)$$

$$\cdot(\mathcal{V}/c)^2\gamma^2. \qquad (6\text{-}138)$$

f $\boldsymbol{E}$ is always equal to minus the
show that this is in agreement
4, which we found in Section 6.5.

is the following:

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J}_m + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right),} \longleftarrow \qquad (6\text{-}139)$$

where $\boldsymbol{J}_m$ is the current density at the point P, where the electric field intensity is $\boldsymbol{E}$ and the magnetic induction is $\boldsymbol{B}$, or

$$\boxed{\oint_C \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 \int_S \left(\boldsymbol{J}_m + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \cdot d\boldsymbol{a} = \mu_0 I_t.} \longleftarrow \qquad (6\text{-}140)$$

The meaning of the index m, for matter, will become apparent later in this section. Let us disregard it for the moment. The current I_t is the total current linking the curve C.

We shall be able to deduce this equation from the fact that Gauss's law is invariant under a Lorentz transformation.

Charges that are stationary at P in reference frame 1, in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are measured, contribute nothing to $\boldsymbol{\nabla} \times \boldsymbol{B}$. Let us disregard them. Imagine that the density of moving charges at P is ρ and that these charges move at a velocity $\mathcal{V}\boldsymbol{i}$. Then

$$\boldsymbol{J} = \rho\mathcal{V}\boldsymbol{i}. \qquad (6\text{-}141)$$

We have chosen the x-axis in the direction of $\boldsymbol{J}$ at P.

In reference frame 2, which follows these charges, $J_{2x} = 0$, and we have a charge density

$$\rho_2 = \frac{\rho}{\gamma} \qquad (6\text{-}142)$$

from Eq. 5-116. In frame 2, $\boldsymbol{B} = 0$, and the only information we have about $\boldsymbol{E}_2$ is Gauss's law:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_2 = \frac{\rho_2}{\epsilon_0} = \frac{\rho}{\gamma\epsilon_0}, \qquad (6\text{-}143)$$

or

$$\frac{\partial}{\partial x_2} E_{2x} + \frac{\partial}{\partial y_2} E_{2y} + \frac{\partial}{\partial z_2} E_{2z} = \frac{\rho}{\gamma\epsilon_0}. \qquad (6\text{-}144)$$

We can deduce an equation for the field in frame 1 by using the equations of transformation of Table 5-10 for the partial derivatives, and those of Table 6-1 for the components of $\boldsymbol{E}$:

$$\gamma\left(\frac{\partial}{\partial x} + \frac{\mathcal{V}}{c^2}\frac{\partial}{\partial t}\right) E_x + \gamma\frac{\partial}{\partial y}(E_y - \mathcal{V}B_z) + \gamma\frac{\partial}{\partial z}(E_z + \mathcal{V}B_y) = \rho/\gamma\epsilon_0, \qquad (6\text{-}145)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} + \frac{\mathcal{V}}{c^2}\frac{\partial E_x}{\partial t} - \mathcal{V}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = \rho\,\frac{(1 - \beta^2)}{\epsilon_0}. \qquad (6\text{-}146)$$

Table 6-3. *Maxwell's Equations*

Differential Form	Integral Form
$\nabla \cdot \boldsymbol{E} = \frac{\rho_t}{\epsilon_0}$	$\int_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_t}{\epsilon_0}$
$\nabla \cdot \boldsymbol{B} = 0$	$\int_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0$
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot d\boldsymbol{a} = -\frac{\partial \Phi}{\partial t}$
$\nabla \times \boldsymbol{B} - \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J}_m$	$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_S \left(\boldsymbol{J}_m + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \cdot d\boldsymbol{a} = \mu_0 I_t$

We have again omitted the subscripts 1. But Gauss's law also applies in frame 1, and $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$. Then, dividing by $\mathcal{V}$,

$$\frac{1}{c^2} \frac{\partial E_x}{\partial t} - \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) = -\frac{\rho \mathcal{V}}{\epsilon_0 c^2} = -\frac{J}{\epsilon_0 c^2} = -\mu_0 J. \tag{6-147}$$

We have arrived at this equation by postulating that the current density vector $\boldsymbol{J}$ was directed along the x-axis. In the more general case where $\boldsymbol{J}$ has three components, we have two other equations that can be deduced from this one by rotating the indices. Combining the three,

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J}_m. \tag{6-148}$$

Since the exact nature of the current is immaterial, we have added a subscript m on $\boldsymbol{J}$ to indicate that it represents any type of current in matter. For example, in a lossy dielectric, $\boldsymbol{J}_m$ is the sum of the conduction current density $\boldsymbol{J}_f$ plus the polarization current density $\partial \boldsymbol{P}/\partial t$.

6.12 MAXWELL'S EQUATIONS

Maxwell's equations are grouped together in Table 6-3. You will be able to show (Problem 6-18) that they are invariant under a Lorentz transformation.

These are the four fundamental equations of electromagnetism. We shall have many occasions to discuss them, and especially to use them, throughout the remaining chapters. We shall not therefore say more about them for the moment.

From: Lorrain & Corson, Electromagnetic Fields And Waves, 2nd ed., 1970, W. H. Freeman and Company