Encyclopedia Polyhedra

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Last Revision: July 20, 2007

Introduction

This book tabulates the numerical data for the polyhedra mentioned and used in R. Buckminster Fuller's books Synergetics and Synergetics 2 as well as other polyhedra which I have found interesting. I have calculated and tabulated this data because I have been unable to find all of this data in any other reference. Other references seem to have bits and pieces of data for some of these polyhedra, often without exact expressions. I have often needed, for example, a particular angle for a particular part of a polyhedron and I have found it a big distraction to have to stop my current train of thought and calculate the needed angle. It is very convenient to have a table of all these polyhedrons' numerical data.

In addition to the polyhedra data, I have collected some useful information about how the polyhedra fit within one another or otherwise are related one to another. This information is presented at the end of each section.

I have found that the so called (by Lynnclaire Dennis) "120 Polyhedron" can define, using its 62 vertices, 10 Tetrahedra, 5 Octahedra, 5 Cubes, 5 rhombic Dodecahedra, 1 Icosahedron, 1 regular Dodecahedron, 1 rhombic Triacontahedron, as well as several "Jitterbugs". For this reason, the coordinates for many of the polyhedra vertices are given in terms of the 62 vertex coordinates of the 120 Polyhedron. The Cartesian (x, y, z) coordinates for these vertices are tabulated in the chapter on the 120 Polyhedron (Type III Dennis). The coordinates are expressed in terms of the Golden Ratio.

Notations and Relations

This is a tabulation of most of the abbreviations used in the book. Other abbreviations are explain in the context where they are used.

The symbol "≅" is to be read as "is approximately equal to."
The symbol "≡" is to be read as "is defined to be."

EL ≡ Edge Length.
FA ≡ Face Altitude.

DFV ≡ Distance from the center of a face to a vertex.
DFE ≡ Distance from the center of a face to a mid-edge point.

DVV ≡ Distance from the center of volume to a vertex.
DVE ≡ Distance from the center of volume to a mid-edge point.
DVF ≡ Distance from the center of volume to a face center point.

See the data for the Tetrahedron for an illustrative use of these abbreviations

The notation DF(V1.V2.V3)E(V2.V3) indicates the distance from the center of the face formed by the vertices V1, V2, and V3, to the mid-edge point along the edge formed by V2 and V3.

In R. Buckminster Fuller's book *Synergetics*, Fuller **defined** the unit edge length regular Tetrahedron to be 1 unit of volume. This is significantly different then the conventional definition of the "unit measure of volume" using the unit edge length Cube. The equations for the volume of the polyhedra are therefore given in terms of the conventional Cube-based unit of measure as well as the Synergetics' Tetra volume unit of measure.

$\varphi \equiv$ Golden Mean, Golden Ratio. "φ" is the Greek letter phi.

$$\varphi = \frac{1+\sqrt{5}}{2} \cong 1.618\ 033\ 989 = \varphi + 0$$

$$\varphi^2 = \frac{3+\sqrt{5}}{2} \cong 2.618\ 033\ 989 = \varphi + 1$$

$$\varphi^3 = 2+\sqrt{5} \cong 4.236\ 067\ 977 = 2\ \varphi + 1$$

$$2\varphi^2 = 3+\sqrt{5} \cong 5.236\ 067\ 977 = 2\ \varphi + 2$$

$$\varphi^4 = \frac{7+3\sqrt{5}}{2} \cong 6.854\ 191\ 966 = 3\ \varphi + 2$$

$$3\varphi^2 = \frac{9+3\sqrt{5}}{2} \cong 7.854\ 101\ 966 = 3\varphi + 3$$

Note that for *n* an integer, $\varphi^{n+1} = \varphi^{n} + \varphi^{n-1}$.

The Fibonacci numbers $f(n)$ is defined by

$$f(n+1) = f(n) + f(n-1)$$

with *n* an integer and $n > 0$ and with $f(0) = 1$, $f(1) = 1$.

It is often mentioned that the Golden Ratio is related to the Fibonacci numbers by the equation

$$\varphi = \lim_{n\to\infty} \frac{f(n+1)}{f(n)}$$

However, this is not unique.

Any number sequence which can be defined by the relation

$$f(n+1) = f(n) + f(n-1)$$

where $f(n)$ is an integer will also have the property that

$$\varphi = \lim_{n \to \infty} \frac{f(n+1)}{f(n)}$$

I have often found the following table of trigonometric relations to be useful.

sin(0°)	=	0	=	cos(90°)
sin(15°)	=	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	=	cos(75°)
sin(18°)	=	$\frac{1}{2\varphi}$	=	cos(72°)
sin(30°)	=	$\frac{1}{2}$	=	cos(60°)
sin(36°)	=	$\frac{\sqrt{3-\varphi}}{2}$	=	cos(54°)
sin(45°)	=	$\frac{1}{\sqrt{2}}$	=	cos(45°)
sin(54°)	=	$\frac{\varphi}{2}$	=	cos(36°)
sin(60°)	=	$\frac{\sqrt{3}}{2}$	=	cos(30°)
sin(72°)	=	$\frac{\sqrt{\varphi+2}}{2}$	=	cos(18°)
sin(75°)	=	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	=	cos(15°)
sin(90°)	=	1	=	cos(0°)

The following relations are also helpful:

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

A system (polyhedron) is defined by its angles. For example, a Tetrahedron is a Tetrahedron regardless of its edge lengths. As you scale a system, its lengths, areas and volume change but its angles all remain the same. A system is therefore characterized by its angles. We should therefore pay particular attention to the polyhedra's angles.

In the Quantum Mechanics' theory of angular momentum, the following "space quantization" equation is used

$$\theta = \arccos\left(\frac{m_j}{\sqrt{j\,(j+1)}}\right)$$

where j is the angular momentum quantum number and m_j is the quantum number giving the projection of the angular momentum onto a preferred axis. The quantum numbers j are positive integers or half-integers. The m_j quantum numbers are restricted to

$$m_j = -j, -j+1, ..., j-1, j$$

This space quantization angle is the angle between the angular momentum vector and the axis defined by the preferred space direction (as can be defined, for example and in one case, by the direction of an externally applied magnetic field.)

For example, if $j = 1/2$ then $m_j = -1/2, 1/2$. For the $m_j = 1/2$ case the space quantization angle θ is

$$\theta = \arccos\left(\frac{\frac{1}{2}}{\frac{1}{2}\sqrt{\frac{1}{2}+1}}\right) = \arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right) \cong 35.264\,389\,68^\circ$$

This is also the half-cone angle for a cone defined by a spinning Tetrahedron. The half-cone angle is defined to be the angle between the surface of the cone and the symmetry (spin) axis of the cone.

The following Table lists several angles using the space quantization angle formula above.

j	m_j	$\theta = \arccos\left(\frac{m_j}{\sqrt{j(j+1)}}\right)$	(90° – θ) [θ – 90°]	180° – θ
1/2	1/2	54.735 610 32°	(35.264 389 68°)	125.264 389 7°
1/2	– 1/2	125.264 389 7°	[35.264 389 68°]	54.735 610 32°
1	1	45°	(45°)	135°
1	0	90°	(0°)	90°
1	–1	135°	[45°]	45°
3/2	3/2	39.231 520 48°	(50.768 479 52°)	140.768 479 5°
3/2	1/2	75.036 782 57°	(14.963 217 43°)	104.963 217 4°
3/2	– 1/2	104.963 217 4°	[14.963 217 43°]	75.036 782 57°
3/2	– 3/2	140.768 479 5°	[50.768 479 52°]	39.231 520 48°

For all convex polyhedra

$$\text{Number of vertices} \times 360° - \text{Sum of surface angles} = 720°$$

Note that the sum of the surface angles of any Tetrahedron (regular or irregular) is 720°. R. Buckminster Fuller interprets this as meaning that to form a closed structure from a plane, you must take out (in surface angle amount) the equivalent of one Tetrahedron. Or, thinking of this in the reverse situation, to make a structure lay flat in a plane, you must add to the structure 720° in surface angle.

For 3-dimensional structures (polyhedra) Euler's equation is

$$\text{Number of Faces} + \text{Number of Vertices} = \text{Number of Edges} + 2$$

$$F + V = E + 2$$

I have found that for the regular polyhedra, which have dual polyhedra, that the sum of the dihedral angle of one polyhedron plus the central angle of the dual polyhedron equals 180°. For example, for the Tetrahedron, the dual polyhedron is also a Tetrahedron (it is its own dual). We have (see the data for the Tetrahedron):

$$\text{dihedral angle} + \text{central angle} = 70.528\ 779\ 366° + 109.471\ 220\ 634° = 180°$$